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AN OPTIMUM MIX OF
HETEROGENEOUS WEAPONS SYSTEMS

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SUMMARY

The objective of the present analysis is to demonstrate the applicability of differential gaming to problems concerning the optimum mix of weapons systems. The problem is formulated as a game between two sides, Blue and Red, over a given period of time. The Blue player, with two weapons systems A and B, desires an optimum mix of the two weapon systems. He then has a tactical allocation over time to the two weapons for an "arsenal" mission and an enemy force "attrition" mission. The Red player, with one weapon system C, has a similar tactical allocation to these two missions. The payoff is given in terms of both sides allocation to the "arsenal" mission.

AN OPTIMUM MIX OF HETEROGENEOUS WEAPONS SYSTEMS

INTRODUCTION

Currently, considerable attention is being given to the problem of determining the optimum mix of heterogeneous weapons systems as exemplified by the cost-effectiveness studies of wings versus divisions and carriers versus submarines with missile weapons systems. The ultimate goal of some of the aforementioned studies is to provide a means for making a trade-off among the respective weapons systems. The complexity of the problem results from the establishing of evaluative criteria for dissimilar entities.

The methodology of this analysis is based on the theory of differential games. In essence it is a two-player problem in the calculus of variations; its one-player counterpart is exemplified by the problems arising in control theory. It is argued that any allocation of weapons systems which does not consider optimal enemy strategies is futile; hence game theory is the obvious choice as the modus operandi. Classical game theory, while delineating the concept of an optimal strategy, has provided an abundance of theorems and techniques which have enjoyed a very limited success in the solving of realistic military problems. In direct contrast, differential gaming has the potential of not only providing a rigorous mathematical analysis of trade-off problems but also provides meaningful solutions to many problems confronting the military decision-maker. In this article, the attribute of a weapon system which dictates its homogenity is its attrition capability on opposing weapons systems. Optimum is defined by the particular payoff employed in the analysis.

AN ILLUSTRATION

The object of the present analysis is to demonstrate the applicability of differential gaming to problems concerning the optimum mix of weapons systems. In order to provide such an illustration, a solution is provided to the problem contained in the following scenario.

Scenario

Two nations, Blue and Red, are engaged in a conflict. Blue has two types of weapons, A and B. Red has one type of weapon, namely C. Weapons A and B destroy type C weapons and type C weapons destroy both weapons types A and B. Each weapon in the particular weapons system can be either in an active status, i.e., destroying opposing weapons or in an arsenal during which time they are not destroying any opposing weapons system; however they are subject to damage from opposing weapons.

Assume that the costs to be assessed each force is a function of the number of weapons it borrows from the arsenal. Therefore the objective of each combatant is to have the maximum number of weapons in his arsenal relative to his opponent. To a degree, each combatant is attempting to fight the war with the least number of active weapons. If either combatant initially allocates all of his weapons to the arsenal, he will be maximizing his payoff in the short-run but not throughout the duration of the conflict since he would not be destroying any attacking weapons.

Assumptions

(1) Type A weapons are the most effective of all weapons.

(2) Each weapon system, for both nations, can be subjected to a given attrition factor regardless of its activity.

(3) The attrition of a friendly unit is directly proportional to the number of engaged opponent units.

(4) Each side knows the values of the opponent's force ratios, supply capability and weapon characteristics.

(5) The time of determination of the conflict (T) is known.

(6) The time necessary to disengage a unit from its current mission to an alternate mission is such that the transition can be accomplished only after a considerable time has expired.

Problem

The problem is to determine an optimum method for allocating each weapons system between an active and arsenal status. Optimum in this case is defined as that allocation which gives Blue the greatest excess of weapons relative to Red throughout the conflict.

State Variables

The following variables give a complete description of a combatant's force throughout the conflict and are defined as state variables.

x_1	Number of type A weapons in arsenal A
x_2	Number of type B weapons in arsenal B
y	Number of type C weapons in arsenal C
T	Time until termination of the conflict

Control Variables

The following variables are those quantities which are under the control of each side and determine the state variables throughout the conflict.

Blue

ψ_1	Fraction of type A weapons to be engaged in direct conflict with the opponent
ψ_2	Fraction of type B weapons to be engaged in direct conflict with the opponent

Red

ϕ	Fraction of type C weapons to be engaged in direct conflict with the opponent

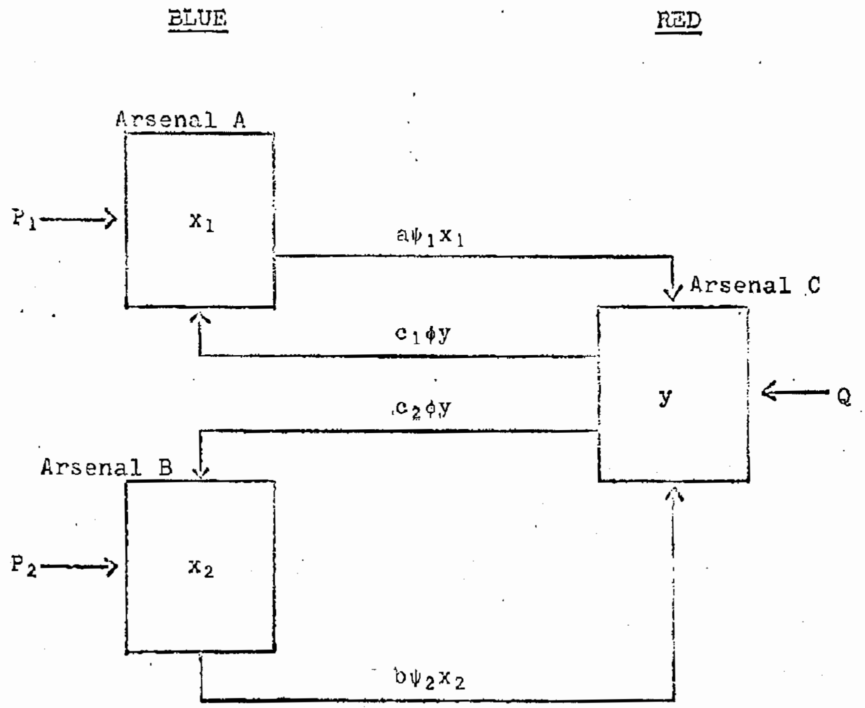
Game Parameters

The following parameters are used to describe the appropriate game units.

a	Attrition factor applied to weapons system C from weapons system A
b	Attrition factor applied to weapons system C from weapons system B
c_1	Attrition factor applied to weapons system A from weapons system C
c_2	Attrition factor applied to weapons system B from weapons system C
P_1	Supply rate of type A weapons
P_2	Supply rate of type B weapons
Q	Supply rate of type C weapons

Schematic

The following is a schematic of the conflict.



Mathematical Analysis

Kinematic Equations

The problem reduces to evaluating the following payoff and constraints.

$$\int_0^T [(1-\phi)y - (1-\psi_1)x_1 - (1-\psi_2)x_2]dt$$

$$\dot{x}_1 = P_1 - c_1\phi y$$
$$\dot{x}_2 = P_2 - c_2\phi y$$
$$\dot{y} = Q - a\psi_1 x_1 - b\psi_2 x_2$$

Since

$$\min_{\psi}\max_{\phi} \Sigma V_i f_i(\vec{x},\phi,\psi) = 0^*$$

the above system reduces to

$$\min_{\psi}\max_{\phi} [(1-aV_3)\psi_1 x_1 + (1-bV_3)\psi_2 x_2 - (1+c_1V_1+c_2V_2)\phi y$$
$$+ P_1V_1 + P_2V_2 + QV_3 + y - x_2 - x_1 - V_T] = 0$$

Letting

$$\psi_1 = \bar{\psi}_1$$
$$\psi_2 = \bar{\psi}_2$$
$$\phi = \bar{\phi}$$

The switching functions are defined as

$$S_1 = 1 - aV_3$$
$$S_2 = 1 - bV_3$$
$$S_3 = -(1 + c_1V_1 + c_2V_2)$$

Hence

$$S_1x_1\bar{\psi}_1 + S_2x_2\bar{\psi}_2 + S_3y\bar{\phi} + P_1V_1 + P_2V_2 + QV_3 + y - x_2 - x_1 - V_T = 0$$

Conditions at a Terminal Surface

$$V = 0$$

$$\frac{\partial V}{\partial s_1} = V_1\frac{\partial x_1}{\partial s_1} + V_2\frac{\partial x_2}{\partial s_1} + V_3\frac{\partial y}{\partial s_1} + V_T\frac{\partial T}{\partial s_1} = V_1 = 0$$

Similarly for V_2 and V_3

Hence

$$S_1 = 1 \qquad \overset{\circ}{V}_1 = S_1\bar{\psi}_1 - 1$$
$$S_2 = 1 \qquad \overset{\circ}{V}_2 = S_2\bar{\psi}_2 - 1$$
$$S_3 = -1 \qquad \overset{\circ}{V}_3 = S_3\bar{\phi} + 1$$

When

$$S_1 > 0 \text{ then } \bar{\psi}_1 = 0$$
$$S_2 > 0 \text{ then } \bar{\psi}_2 = 0$$
$$S_3 < 0 \text{ then } \bar{\phi} = 0$$

* Isaacs, Rufus, Differential Games, Wiley, 1965, p 67

The First Transition Surface

$$\overset{\circ}{V}_1 = -1 \qquad \text{since:} \quad \overline{\psi}_1 = 0$$
$$\overset{\circ}{V}_2 = -1 \qquad \qquad \overline{\psi}_2 = 0$$
$$\overset{\circ}{V}_3 = 1 \qquad \qquad \overline{\phi} = 0$$

Solving the above equations subject to the following initial conditions:

$$T = 0$$

$$V_1 = V_2 = V_3 = 0$$

And substituting into the switching functions gives:

$$T = \frac{1}{a} \qquad T = \frac{1}{b} \qquad T = \frac{1}{c_1 + c_2}$$

The first transition surface occurs when:

$$(1) \quad T = \frac{1}{a} = t$$

The Second Transition Surface

$$\overset{\circ}{V}_1 = -aV_3 \qquad \text{since:} \quad \overline{\psi}_1 = 1$$
$$\overset{\circ}{V}_2 = -1 \qquad \qquad \overline{\psi}_2 = 0$$
$$\overset{\circ}{V}_3 = 1 \qquad \qquad \overline{\phi} = 0$$

Solving the above equations subject to the following initial conditions:

$$T = \frac{1}{a}$$
$$V_1 = -T$$
$$V_2 = -T$$
$$V_3 = T$$

And substituting into the switching function gives:

$$T = \frac{1}{a} = t$$

$$T = \frac{1}{b} = t_0'$$

$$(2) \quad T = \frac{-c_2 + \sqrt{c_2^2 + 2c_1 a - c_1^2}}{c_1 a} = t_0$$

With:

$$\overline{\psi}_1 = 1$$
$$\overline{\psi}_2 = 0$$
$$\overline{\phi} = 1$$

Assuming $t_0 < t_0'$. The case $t_0 > t_0'$ can be solved in a like manner.

The Third Transition Surface

$$\dot{V}_1 = -aV_3 \qquad \text{since:} \quad \bar{\psi}_1 = 1$$
$$\dot{V}_2 = -1 \qquad \bar{\psi}_2 = 0$$
$$\dot{V}_3 = -c_1V_1 - c_2V_2 \qquad \bar{\phi} = 1$$

Solving the above equations subject to the following initial conditions:

$$V_1 = -\frac{1}{2}(a\tau^2 + \frac{1}{a})$$
$$V_2 = -\tau$$
$$V_3 = \tau$$
$$\tau = t_0$$

And substituting into the switching function gives:

$$(3) \quad T = t_0 + \frac{\ln}{\lambda}\left[\frac{H + \sqrt{H^2 + 4L_1L_2}}{2L_1}\right]$$

Where:

$$L_1 = -\frac{1}{4}at_0^2 - \frac{1}{2}t_0\left(\frac{c_2}{c_1} + \frac{a}{\lambda}\right) - \frac{1}{4a} - \frac{c_2}{2c_1\lambda}$$
$$L_2 = -\frac{1}{4}at_0^2 - \frac{1}{2}t_0\left(\frac{c_2}{c_1} - \frac{a}{\lambda}\right) - \frac{1}{4a} + \frac{c_2}{2c_1\lambda}$$
$$\lambda = \sqrt{ac_1}$$
$$H = \frac{a}{\lambda}\left(\frac{1}{b} - \frac{c_2}{c_1a}\right)$$

With:

$$\bar{\psi}_1 = 1$$
$$\bar{\psi}_2 = 1$$
$$\bar{\phi} = 1$$

Results

The proceding analysis offers the following results:

(1) The optimum mix of type A weapons to type B weapons can be determined by calculating the ratio of the control variables ψ_1/ψ_2 and $(1-\psi_1)/(1-\psi_2)$.

The former represents the ratio of the fractions of type A weapons to type B weapons in direct attack and the latter represents the ratio of the fractions of type A weapons in the arsenal to type B weapons in the arsenal. In the current scenario these ratios are either zero or one.

(2) The war begins with all weapons in use and ends with all surviving weapons in their respective arsenal.

(3) The force with two weapons systems (Blue) will place the lesser effective of the weapons systems, (B), i.e., the one with the smaller of the two attrition factors, in the arsenal prior to taking a comparable action for the more effective weapon system. The optimal times for Blue to place type A weapons in the arsenal and type B weapons in the arsenal are given by equations (1) and (3) respectively.

(4) The optimal time for Red to place his weapons in the arsenal is given by equation (2).

(5) The value of the game, i.e., the difference between the number in Blue's and Red's arsenal can be determined by integrating the appropriate path equations between the transition surfaces.

The Limitations of the Model and the Results

Criticisms of the model fall into two general categories: criticisms of the assumptions, called Type I; and criticisms of the techniques, called Type II. The importance of segregating the criticisms into different types is that the limitations of the results attributed to the assumptions should not be construed as limitations of the *modus operandi*.

restriction can be overcome by changing the kinematic equations. In the current model, it was assumed that the attrition of a combat weapon unit was proportional only to the number of engaged opponent weapons units. Hence, $c_1\phi y$ represents a fixed numerical loss of each type A weapon from each type C weapon. If $c_1\phi y$ were replaced by $c_1\phi x_1 y$, then each Red weapon would be attriting a fraction of type A weapons rather than a fixed number of A weapons. A modification of the kinematic equations as previously suggested will yield strategies which result in fractional allocation of weapons to the arsenal.

(2) Modification as discussed above would also result in the decisions being a function of the supply rates. In the current model, the difference between Blue and Red weapons in their arsenals at any time is a function of the supply rates but the time to switch the units to the arsenal is independent of the supply rates.

(3) Another limitation concerns the measurability of model parameters; i.e., of the attrition factors a, b, c_1 and c_2. The parameter c_1 represents the reduction of Blue's type A weapons from one Red weapon; i.e., one type C weapon can reduce c_1 of Blue's type A weapons per unit time. An attrition factor such as c_1 is a function of many quantities such as firepower, posture, defense capability and mobility. By no means is the difficulty in obtaining these parameters to be underestimated. However, problems of this nature are present in most war gaming problems and have not stymied efforts in this direction. One possible solution to the problem is to use existing simulations to provide inputs to differential games.

his losses or maximize enemy casualties or some other objective function of military significance. This limitation emphasizes the need for constructing many differential models under different assumptions with a spectrum of military objectives.

(5) The present model assumes that the time T of termination of the conflict is known; this problem may be overcome in several ways. First, one may maximize or minimize some function at the end of the war quite independent of the exact time of termination. One will recall that the present model maximized the number of weapons in the arsenals relative to the enemy during the conflict, not at time of termination. Mathematically speaking, one may work with a terminal payoff in lieu of an integral payoff. Another solution is to calculate the strategies for different times of termination. This presents no difficulty since the strategies are a function of a state variables.

(6) The reader will note that there are no constraints which restrict the state variables to the positive quadrant; i.e., prevent the number of weapons from becoming negative. It can be shown that these additional constraints add nothing of military significance to the existing solution since when the opposing force is reduced to zero, it is irrational to devote an increasing fraction of weapons to destroy opposing weapons.

<u>Type II Limitations</u>

Type II limitations present greater difficulties to overcome than those encountered in the discussion above. Published works in the field of differential gaming are rather sparse; the first determine whether or not a solution exists before attempting the analysis. Furthermore, the problems are so diverse in nature that the derivation of any such theorems may be of little significance. However, we may argue that an analysis which results in no solution can in some circumstances yield meaningful information. For example, a rigorous statement, in differential gaming terminology, of some of the existing models of military conflicts may show the impossibility of obtaining strategies. It is argued that although no solution exists an analysis of this nature does render useful information in that it provides a method for discovering the futility of certain approaches.

(2) Presently, the theory of differential games assumes perfect information; i.e., both sides are assumed to have complete knowledge of all the variables and parameters; e.g., force size, attrition factors, and supply rates. Such an assumption precludes any solution in mixed strategies; i.e., the case where each combatant has several strategies available to him and chooses them according to a probability distribution. Unfortunately, there are few arguments that one may present to reduce the limitations imposed by this assumption. The only optimistic conjecture that one may state is that additional research may provide insights into this problem; the results of preliminary investigations toward a theory of incomplete information have not been promising.

programs, the general analysis is of a complexity which usually precludes routinization. Hence, any meaningful military problem which is to be solved by differential gaming will usually involve a lengthy analysis.